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GENERAL SOLUTION OF THE DYNAMICS OF A VISCOELASTIC IMPACT BETWEEN A SPHERICAL BODY AND A PLANE SURFACE OF SEMI-SPACE

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ABSTRACT

The brief analysis the problems of an elastic and a viscoelastic contact have been given. The general solution of the dynamics of viscoelastic impact between a spherical body and a plane surface of semi-space at an arbitrary angle of attack has been examined in this paper. Solutions to problems of the normal and the tangential displacement have been considered. The differential equations of the displacement (the movement) of the centre of mass of the body have been obtained and their approximate solutions have been found. Also, the boundaries of application of the obtained equations of displacement for the viscoelastic contact have been given.

KEYWORDS: viscoelastic impact, parameters viscoelasticity, viscoelastic forces, dynamic modules, method specific forces, equivalent work, restitution, compression, the motion of rolling shear, movement, motion differential equations, the headway (the translational) motion under the deformation of direct shear.

1. INTRODUCTION

The mechanics of an elastic and a viscoelastic contact problems between two smooth surfaces have been studied in the 19-th century by Herts [1] and Boussinesq [2], and then later, for example, it was examined by many others researchers, such as: Bowden and Tabor [3]; Landau and Lifshits [4]; Mindlin [5];, Timoshenko and Goodier [6]; Archard [7]; Radok [8]; Goldsmith [9]; Hunter, [10]; Galin [11]; Lee [12]; Lee and Radok [13]; Graham [14]; Sneddon [15]; Greenwood and Williamson[16]; Ting [17]; Simon [18]; Johnson, Kendall and Roberts [19]; Derjaguin, Muller and Toporov [20]; Bush, Gibson and Thomas [21]; Moore [21,22];Maw, Barber and Fawcett [23]; Tabor [24]; Cundall and Strack [25]; Jonas [26]; Padovan and Paramadilok [26]; Johnson [27]; Schafer, Dippel and Wolf [28]; Brilliantov, Spahn, Hertzsch, Poeschel [29]; Ramírez, Poeschel, Brilliantov, Schwager [30]; Stronge [31]; Barber and Ciavarella [32];Laursen [33]; Cheng, Xia, Scriven and Gerberich [34]; Bordbar and Hyppänen [35]; Schwager and Poschel [36,37]; Becker, Schwager and Pöschel [38]; Carbone, Lorenz, Persson and Wohlers [39]; Persson [40]; Cummins, Thornton and Cleary [41];Popov [42, 43]; Heβ [44]; Thornton and Yin [45]; Menga, Putignano, Carbone, and Demelio [46];Popov and Heβ [47]; Lyashenko and Popov [48]; Goloshchapov [49, 50, 51];Mindlin and Deresiewicz [52].According to all these researches, we can allocate several main types of frictional contact:

- Elastic contact (in nature practically doesn't exist), when only the elastic forces work between the contacting surfaces.
- Viscoelastic contact, when between the surfaces, which are in contact, the dissipative forces of viscosity (also named the forces of internal friction) begin to act as well.
- The elastic-plastic contact, when forces of viscosity are considerable and the contacting surfaces pass into a plastic state.
- Adhesive contact, when significant adhesive forces act between pure surfaces being in contact, for example under intensive sliding loading, when films absorbed on the contacting surfaces are destroyed, or for example in process friction in a deep vacuum. Thus, in a general case of impact we can neglect adhesive molecular forces and do not consider its influences here.

But, as we know, in practice, the real contact between contacting surfaces usually is implemented as a combination of these four basic types of contact. On the other hand, we can allocate three main types of contact of the relative displacement between the contacting surfaces, such as a slip, a rolling motion and an impact. But indeed, as we will see soon, the impact includes two independent tangential deformations, such as the shear under a rolling motion (the deformation of rolling shear) and the deformation of direct headway shear.

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Let a spherical body, having the mass m and the radius $\boldsymbol{R}$, and the translational velocity $\boldsymbol{V_0}$ relative to the still semi-space, and the relative angular velocity of rotation $\boldsymbol{\omega_{zi}}$ around the moving centre of mass of a body in the plane $X0Y \equiv XAY$, comes into contact under the angle of attack $\boldsymbol{\alpha}$ to the surface of semi-space at the initial instant of the time $t = 0$, as it is depicted in the Figure 1, and also let the angular velocity of the centre of mass of a body relative to the still semi-space equals zero.

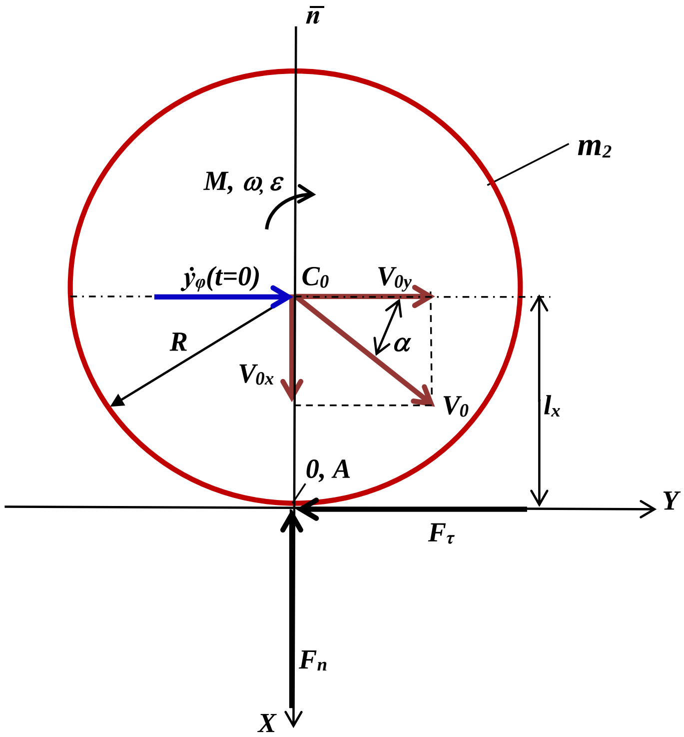

Figure 1. Schematic illustration of the mechanics of a viscoelastic contact between a spherical solid body and a semi-space at impact.

Also, on the other hand, since the angular velocity of the body relative to the centre of mass of the body equals $\boldsymbol{\omega_{zi}}$, consequently, it is obvious that all points of surface of a body in the plane XAY have the relative circular velocity $\boldsymbol{\omega_{zi}} \times \boldsymbol{R}$ and since the point $\boldsymbol{A}$ is the instantaneous centre of rolling velocities in the plane XAY (the centre of rolling velocities always is placed in the point $\boldsymbol{A}$), therefore, it is obvious, that in the initial of moment of the time of contact t=0 the body starts to roll in surface of semi-space in the plane XAY with the initial velocity of rolling $\dot{\boldsymbol{y}}_{\boldsymbol{\varphi}}\boldsymbol{(t=0)= \omega_{zi} \times R}$. Also, in the Figure 1. are designated: $\bar{\boldsymbol{n}}$ is the normal to the surface; $V_{0x} = V_0 \sin\alpha$, $V_{0y} = V_0 \cos\alpha$ are respectively the normal and tangential velocities of the centre of mass of a body right before to the beginning of an impact; $M = F_\tau\, l$ is the rotational moment and where $\boldsymbol{l = R}$ is the radius of the body and therefore it can be taken as the shoulder of tangential force; ω is the angular velocity; $\varepsilon = \ddot{\varphi}$ is the angular acceleration of a body around the centre of mass of a body and where φ is the angular rotation of a body around the centre of mass of a body. Also, it is taken here that the reactive tangential viscoelastic force is applied in the point A, and in the initial moment of the time the point of contact $\boldsymbol{A}$ lays in the point $\boldsymbol{0}$ of the beginning of the coordinates.

The illustration of the displacement of a body in the surface of semi-space in the current moment of the time of impact is depicted as well in the Figure 2., where: x is the distance of the mutual approach or the total deformation of compression between surfaces of the colliding bodies, and as well, in the same of instant of the time, it is the displacement of centre of mass of a body on the axis coordinate $\boldsymbol{X}$; $\boldsymbol{V_d}$ is the volume of deformations, which is forming in the course of contact.

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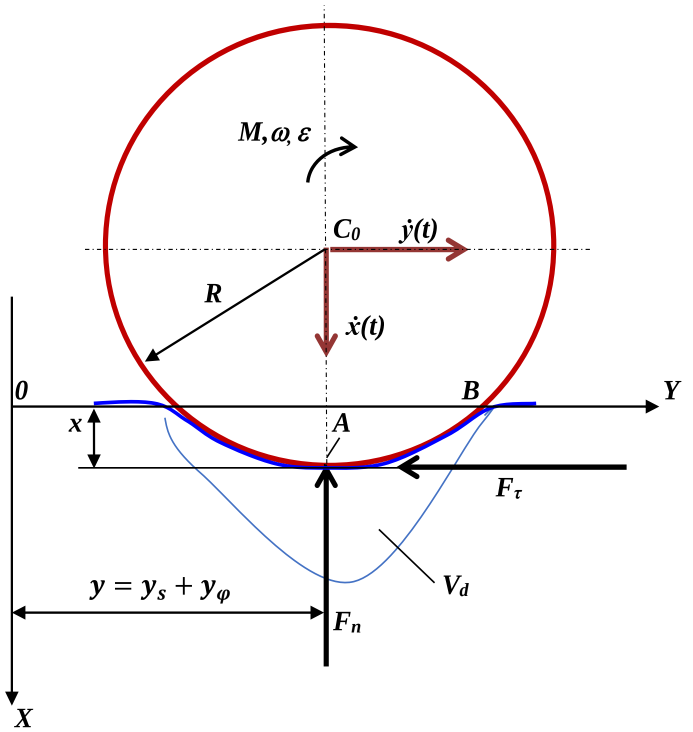

Figure 2. The illustraion of the displacement of a body in the surface of semi-space in the current moment of the time of impact

It is obvious that the viscoelastic forces $\boldsymbol{F_n}$ and $\boldsymbol{F_\tau}$ are acting in the contact area between the surfaces of contact and according to Newton's Second Law we can write the next differential equations of motion:

$$m\ddot{x} = -F_n, \tag{1}$$

$$m\ddot{y} = -F_\tau, \tag{2}$$

where m is the effective mass or reduced mass, which in this case is equal to the mass of the spherical body; $\ddot{y}, \ddot{x}$ are the accelerations of the centre of mass of a body.

On the other hand, since the rotational moment of $\boldsymbol{M}$ and the angular acceleration $\varepsilon = \ddot{\varphi}$ act in the direction of rotation of spherical body, and therefore they are taken positive, or in others words, the rotational moment of force of friction is positive as its direction coincides with the direction of positive counting of angle of rotation φ, and thus, the equation for the rotational motion can be written as

$$J_z\ddot{\varphi} = M = RF_\tau, \tag{3}$$

where J_z is the effective moment of inertia which in this case is equal to the moment of inertia of the spherical body.

***Remark**: The equation (3) is totally correct, like it is usually taken, for example in [53], [54], [55] and in many others books. Moreover, for example, it was taken in the work [56] that angular momentum $P_\omega = J_z d\omega = RF_\tau dt$, but since the rotational moment $M = P_\omega/dt$, we get the same result as in Eq. (3), namely $J_z\ddot{\varphi} = RF_\tau$.

Also, the first problem , which we have to solve here is the definition of the normal and the tangential forces. There were attempts in the past to use some linear models for solutions of the contact's problems, for example, similar as in the linear Winkler Foundation Model of an elastic foundation, see for example, Kerr [58], it is assumed that the contact forces are proportional to the deflection x and y: $F_n = k_x x$, $F_\tau = k_y y$, where the

coefficients k_x and k_y are called as the modulus of foundation. Or other scholars, see for example [5], [48], take coefficients k_x and k_y (or stiffnesses), as the constant magnitudes too. But it is well known that, these models give an oversimplified and inadequate description of the local contact interaction between curvilinear surfaces of two solid bodies and lead to erroneous conclusions, because they can be used only for linear deformations. The main problem is that all viscoelastic forces are not the linear functions relative to displacements *x*, *y* and *z*. But indeed, it is known, see for example Herts [1], Landau and Lifshits [4]; Brilliantov et al. [29], Goloshchapov[49, 50, 51]; Johnson [27]; Popov [42, 43]; Heβ [44], the parameters (coefficients) k_x and k_y are the variable functions depending of the displacements (motions) *x* and *y*. Also, the most basic problem in the finding of solutions for equations (1,2,3) is that the dynamic contact between two curvilinear surfaces is a non-equilibrium, a nonlinear process of deformations and as well that all mechanical dynamic parameters of viscoelasticity are not the constant values. They all are variable magnitudes, because all dynamic mechanical and physical properties of materials depend on dynamic conditions of loading (displacements, velocities and a frequencies) and temperature. The methods for the finding of viscoelastic parameters and viscoelastic forces, which are using in this paper, you can find in the works [49, 50, 51].
Further, the comparison of Eqs. (2) and (3) gives

$$\frac{1}{R} J_z \ddot{\varphi} = -m\ddot{y} \tag{4}$$

Since for a spherical body $J_z = \frac{2}{5} mR^2$, we get

$$\frac{2}{5} R\ddot{\varphi} = -\ddot{y} \tag{5}$$

Hence, it is obvious that, the common tangential motion (or displacement) ***y*** of the centre of mass of a body can be found as the sum of two motions: the rolling motion y_φ plus the headway (the translational) motion under the deformation of direct shear y_s, namely

$$y = y_\varphi + y_s\,,\; \dot{y} = \dot{y}_\varphi + \dot{y}_s\,,\;\; \ddot{y} = \ddot{y}_\varphi + \ddot{y}_s\,, \tag{6}$$

***Remark**: It is very important to understand here that at viscoelastic impact, the headway (the translational) motion under the deformation of direct shear $\boldsymbol{y_s}$ is not a slip between contacting surfaces – it is specific deformation with the internal friction.
Further, since the rolling motion $y_\varphi = R\varphi$, it follows $\dot{y}_\varphi = R\dot{\varphi}$ and $\ddot{y}_\varphi = R\ddot{\varphi}$, we get the equation, which connects the translational rolling acceleration with the total translational acceleration of body's centre of mass, as

$$\ddot{y}_\varphi = -\frac{5}{2}\ddot{y} \tag{7}$$

As we can see from this equation that the common acceleration of centre of mass of a body $\ddot{y}$ is not equal to the rolling (rotational) acceleration $\ddot{y}_\varphi = R\ddot{\varphi}$, and it is obvious that the common displacement of centre of mass of a body y will not be equal to the rolling displacement y_φ.
***Remark:** It is necessary to mark as well here that some scholars take the rotational moment negative $M = -RF_\tau$, see for example [49], [51], [58]. In this case the equation for rotational motion is written as $J_z\ddot{\varphi} = -RF_\tau$, but as well in their solutions they take that $\dot{y}_\varphi = -R\dot{\varphi}$, and as well it is obvious that $\ddot{y}_\varphi = -R\ddot{\varphi}$, and then, after substitutions we get same equation like we already obtained in the Eq. (7). Thus, it confirms once more that the Eq. (7) is totally exact reflects the motion of body under actions of the frictional force and of the rotational moment of the force of friction.
Then, according to Eqs. (6) it follows

$$\ddot{y}_s = \frac{7}{2}\ddot{y}; \tag{8}$$

$$\ddot{y}_s = -\frac{7}{5}\ddot{y}_\varphi \tag{9}$$

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As we can see, the acceleration the headway (the translational) motion under the deformation of direct shear is stronger than the acceleration of rolling motion.
The integration of the Eq. (7) gives $\dot{y} = \frac{2}{5}\dot{y}_\varphi + C$. Further, since according to the initial conditions $t=0$, $\dot{\varphi}(0) = \omega_{zi}$, $\dot{y}_\varphi(0) = R\dot{\varphi}(0) = R\omega_{zi}$, $\dot{y}_s(0) = V_{0y}$ and as well it is obvious that $\dot{y}(0) = V_{0y} + R\omega_{zi}$, it follows that $C = V_{0y} + \frac{7}{5}R\omega_{zi}$. Hence finally, the solution of the Eq. (7) can be written as

$$\dot{y} = V_{0y} - \frac{2}{5}\dot{y}_\varphi + \frac{7}{5}R\omega_{zi}, \tag{10}$$

or

$$\dot{y}_\varphi = \frac{5}{2}V_{0y} - \frac{5}{2}\dot{y} + \frac{7}{2}R\omega_{zi} \tag{11}$$

The initial angular velocity of rotation $\boldsymbol{\omega_{zi}}$ should be taken positive, when it gives the motion in same direction as the initial translational velocity $\boldsymbol{V_{0y}}$, and it should be taken negative when the rotation goes in the back direction.
Remark: Also, in the paper [48] it was taken in the initial conditions for the translative motion that $\dot{y}(0) = V_{0y}$. It is incorrect choice, because there it was not taken in account that the circular motion in the point A in the initial time of contact becomes the translational motion of the centre of mass a body. This is obvious that the initial total translational velocity of the centre of mass of a body is the sum of two velocities, like it is taken in this article above.
After the integration of the Eqs. (10) and (11) with the initial conditions t = 0, y = 0, yφ = 0, ys = 0we get respectively

$$y = V_{0y}t - \frac{2}{5}y_\varphi + \frac{7}{5}R\omega_{zi}t \tag{12}$$

and

$$y_\varphi = \frac{5}{2}V_{0y}t - \frac{5}{2}y + \frac{7}{2}R\omega_{zi}\,t, \tag{13}$$

Now, using Eqs. (6), we can write the next series of useful equations for the velocities of motions, such as:

$$\dot{y} = \frac{5}{7}V_{0y} + \frac{2}{7}\dot{y}_s + R\omega_{zi} \tag{14}$$

and

$$\dot{y}_s = \frac{7}{2}\dot{y} - \frac{5}{2}V_{0y} - \frac{7}{2}R\omega_{zi}, \tag{15}$$

as well

$$\dot{y}_\varphi = \frac{5}{7}V_{0y} - \frac{5}{7}\dot{y}_s + R\omega_{zi} \tag{16}$$

and

$$\dot{y}_s = V_{0y} - \frac{7}{5}\dot{y}_\varphi + \frac{7}{5}R\omega_{zi}\,. \tag{17}$$

And also, we can write the series of useful equations for the motions, respectively as:

$$y = \frac{5}{7}V_{0y}t + \frac{2}{7}y_s + R\omega_{zi}t \tag{18}$$

and

$$y_s = \frac{7}{2}y - \frac{5}{2}V_{0y}t - \frac{7}{2}R\omega_{zi}t, \tag{19}$$

and as well

$$y_\varphi = \frac{5}{7}V_{0y}t - \frac{5}{7}y_s + R\omega_{zi}t \tag{20}$$

and

$$y_s = V_{0y}t - \frac{7}{5}y_\varphi + \frac{7}{5}R\omega_{zi}t\,. \tag{21}$$

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2. SOLUTION PROBLEMS OF THE NORMAL DISPLACEMENT

In the papers [49], [50] and [51] have been proposed only partial solutions of the equations (1) and (2). But, on the other hand, it is obvious that the general solution can be derived by using the equations (1-21) above and also by using the equations for viscoelastic forces, which have been obtained in the papers [49, 50, 51], by using the methods "MSF" and "MDSF".

According to [49, 50, 51], the expressions for the normal elastic force F_{cn} and for the normal viscous force F_{bn} can be written as follows:

$$F_{cn} = \frac{4}{3}E'R^{1/2}x^{3/2}, \tag{22}$$

$$F_{bn} = 4k_p\frac{E''}{\omega_x}R^{1/2}\dot{x}x^{1/2}, \tag{23}$$

where $E'$is the effective dynamic elasticity module at the compression, $E''$is the effective viscosity modulus, ω_x is the frequency of damped oscillations by axis X.

***Remark:** The dynamic modulus of elasticity is called also as accumulation or storage modulus, and the dynamic modulus of viscosity is called also as the loss modulus.

And since $F_n = F_{bn} + F_{cn}$ the equation for the general normal viscoelastic forces can be written

$$F_n = 4k_p\frac{E''}{\omega_x}R^{1/2}\dot{x}x^{1/2} + \frac{4}{3}k_pE'R^{1/2}x^{3/2} \tag{24}$$

According to Eqs. (1) and (24) the differential equation of the movement (displacement) of the centre of mass of a body by axis X can be expressed as

$$m\ddot{x} + 4k_p\frac{E''}{\omega_x}R^{1/2}\dot{x}x^{1/2} + \frac{4}{3}k_pE'R^{1/2}x^{3/2} = 0, \tag{25}$$

or it can be also written in the canonical form as

$$m\ddot{x} + b_x\dot{x} + c_x x = 0, \tag{26}$$

where the expressions for the variable viscoelasticity parameters can be written as:

$$c_x = \frac{4}{3}E'R^{1/2}x^{1/2}, \tag{27}$$

$$b_x = \frac{4E''R^{1/2}}{\omega_x}x^{1/2}. \tag{28}$$

For practical application of the differential equation (26) with the variable viscoelasticity parameters, we can find their approximate solutions in the same manner as for the equations with the equivalent constant viscoelasticity parameters, if we choose the equivalent constant parameters B_x, C_x so that the work A_{xcm} and A_{xbm} with the variable viscoelasticity parameters c_x, b_x will be equal to the work with the constant viscoelasticity parameters, like it was made, using the method of the equivalent works in [49, 50, 51]. Thus, according to the boundary conditions $t = \tau_1$, $x = x_m$, where τ_1 is the period time of the compression, xm is the maximum magnitude of the compression between surfaces of the contacting bodies (also, as we already know, it is the maximum displacement of the centre of mass of a body, which is equal to the maximum of mutual approach between bodies, and also using the known expressions for work A_{xcm} and A_{xbm} , see [49, 50, 51], we can write next equations:

$$A_{xcm} = C_x\int_0^{x_m} x\,dx = \frac{1}{2}C_x x_m^2 = \frac{8}{15}k_pE'R^{1/2}x_m^{5/2} \tag{29}$$

and

$$A_{xbm} = B_x\int_0^{x_m}\dot{x}\,dx = B_x\frac{\int_0^{x_m}xdx}{\int_0^{\tau_1}dt} = B_x\frac{x_m^2}{2\tau_1} = \frac{8k_pE''R^{\frac{1}{2}}x_m^{\frac{5}{2}}}{5\omega_x\tau_1}. \tag{30}$$

Hence, according to the results obtained in Eqs. (29), (30), we can write the expressions for the equivalent constant viscoelasticity parameters, respectively as:

$$C_x = \frac{16}{15} k_p E' R^{1/2} x_m^{1/2}, \tag{31}$$

$$B_x = \frac{16E'' k_p R^{1/2}}{5\omega_x} x_m^{1/2} \ . \tag{32}$$

Thus, the equation (23) with variable parameters can be rewritten as the equations with constant parameters as follows

$$m\ddot{x} + B_x \dot{x} + C_x x = 0 \tag{33}$$

The equation (33) is the equation of damped oscillations and the solution to this equation with the initial condition t = 0, $\dot{x} = V_{0x}$, is known as

$$x = \frac{V_x}{\omega_x} e^{-\delta_x t} \sin(\omega_x t) \tag{34}$$

and

$$\dot{x} = \frac{V_{0x}}{\omega_x} e^{-\delta_x t} [\omega_x \cos(\omega_x t) - \delta_x \sin(\omega_x t)] \tag{35}$$

Where: $\omega_x = \sqrt{\omega_{0x}^2 - \delta_x^2}$; $\delta_x = \frac{B_x}{2m}$ is the normal damping factor; $\omega_{0x} = \sqrt{\frac{C_x}{m}}$ is the angular frequency of free harmonic oscillations by axis X.
Obviously, the period time of the compression τ_1 can be find from the conditions $\dot{x} = 0$ and $t = \tau_1$ as

$$\tau_1 = \frac{1}{\omega_x} arctan(\omega_x / \delta_x) \tag{36}$$

Also, the expression for the maximum magnitude of the compression between a body and a semi-space has been derived in papers [49, 50, 51], as

$$x_m = \left[\frac{15 m V_{0x}^2}{16 k_p E' R^{1/2}} k_x \right]^{2/5}, \tag{37}$$

where k_x is the restitution coefficient.
In the case of totally elastic impact, when $k_x = 1$ and $k_p = 1$ we get the same result, as it has been obtained by L. Landau [4] according to the Hertz theory [1] for the absolutely elastic contact.
Also other equations, which are using in this article, have been obtained in [49, 50, 51], such as:

$$k_x = \frac{\tau_1}{\tau_2}, \tag{38}$$

where τ_2 is the period of the time of restitution

$$k_x = \frac{(\pi - 3tg\beta_E)}{(\pi + 3tg\beta_E)} \tag{39}$$

and as well

$$tg\beta_E = \frac{\pi}{3} \times \frac{(1 - k_x)}{(1 + k_x)}, \tag{39'}$$

$$tg\beta_E = \frac{E''}{E'} = \frac{E_1'' E_2'' (E_1' + E_2')}{E_1' E_2' (E_1'' + E_2'')}, \tag{40}$$

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where β_E is the effective angle of mechanical losses for the normal displacement, see [51].
Finally, from Eqs. (37) and (39) it follows that

$$x_m = \left[\frac{15mV_x^2}{16k_p E' R^{1/2}} \times \frac{(\pi - 3tg\beta_E)}{(\pi + 3tg\beta_E)}\right]^{2/5} \tag{41}$$

Also, using Eq. (35) with the boundary conditions $t = \tau_x$, $V_{tx} = \dot{x}(\tau_x)$, (V_{tx} is the normal velocity of centre of mass of a body in the instant of rebound), the duration of the time of impact $\tau_x = \tau_1 + \tau_2$, can be found, as

$$\tau_x = -\frac{\ln k_x}{\delta_x}, \tag{42}$$

where

$$\delta_x = \frac{B_x}{2m} = \frac{8k_p E'' R^{1/2}}{5m\omega_x} x_m^{1/2} = \frac{8k_p E' tg\beta_E}{5\pi m} \tau_x R^{1/2} x_m^{1/2} \tag{43}$$

and finally by using Eq. (37), (39) and (41), (43) , see as well [51], we get

$$\tau_x^2 = -\frac{2(1+k_x)\ln k_x}{V_{0x}^{2/5}(1-k_x)k_x^{1/5}} \times \left(\frac{5m}{8k_p E' R^{1/2}}\right)^{4/5} \tag{44}$$

3. SOLUTION PROBLEMS OF THE TANGENTIAL DISPLACEMENT

The expressions for the tangential elastic force F_{cs} and for the tangential viscous force F_{bs} for the headway (the translational) motion under the deformation of direct shear y_s already have been obtained by using the methods "MSF" and "MDSF", see [49, 50, 51], can be expressed as follows

$$F_{cs} = G' P_x y_s \tag{45}$$

and

$$F_{bs} = \frac{G''}{\omega_y} P_x \dot{y}_s, \tag{46}$$

where $P_x = k_h x + 2k_p R^{1/2} x^{1/2}$, $G'$is the effective dynamic elasticity module at the shear, $G''$is the effective viscosity modulus at shear and where k_h isthe coefficient of the depth of the contact surface, ω_y is the frequency of damped oscillations by axis Y.
On the other hand, as it is known [51], [53], [54], [58] that, in the case of rolling between the contacting surfaces, the instantaneous centre of rolling velocities always is placed in the plane XAY in the point $\boldsymbol{A}$, see Figure 2. Therefore, it is obvious, that the velocity of rolling motion in the point $\boldsymbol{A}$ always equals zero, $\dot{y}_{\varphi A} = 0$, but the velocity in any point of the surface of contact $\dot{y}_{\varphi i} = \dot{y}_{ri}$ is not equal zero, we can write for the angular velocity of the relative rotation between the colliding bodies, see Fig. 2., that

$$\omega = \frac{\dot{y}_{ri}}{(x - x_i)} = \frac{\dot{y}_\varphi}{R}, \tag{46'}$$

where x_i is coordinate of any point on the surface of contact and as well, in the same time, it is the indentation of any point of the body into a semi-space, $\dot{y}_{ri}$is the velocity of the tangential deformation of the rolling shear in any point of the surface of contact.
On the other hand, since, it is obvious that for all points of the surface of contact, including the point B in the plane $Z0X$, $x_i = 0$, see Fig. 2., the velocity of the tangential deformation of the rolling shear can be found as

$$\dot{y}_r = \frac{x}{R} \dot{y}_\varphi \tag{47}$$

We can rewrite the equation (47) as $dy_r = \frac{x}{R} dy_\varphi$, but since y_φ and x are linearly independent, the tangential deformation at shear in the case of rolling contact can be expressed as

$$y_r = \frac{x}{R} y_\varphi \tag{48}$$

Now, according to Eqs. (47), (48), the expressions for the tangential elastic force F_{cr} and for the tangential viscous force F_{br} for the motion of the rolling shear y_φ, can be expressed as

$$F_{cr} = \frac{x}{R} G' P_x y_\varphi, \tag{49}$$

$$F_{br} = \frac{G''}{\omega_y} \frac{x}{R} P_x \dot{y}_\varphi \tag{50}$$

Since the general tangential force can be expressed as sum $F_\tau = F_{cs} + F_{bs} + F_{cr} + F_{br}$, we get

$$F_\tau = G' P_x y_s + \frac{G''}{\omega_y} P_x \dot{y}_s + \frac{x}{R} G' P_x y_\varphi + \frac{G''}{\omega_y} \frac{x}{R} P_x \dot{y}_\varphi \tag{51}$$

Now substituting y_φ and $\dot{y}_\varphi$ from Eqs (16) and (20) into Eq (51) and then after simple algebraic actions, we obtain

$$F_\tau = G' q_x y_s + \frac{G''}{\omega_y} q_x \dot{y}_s + u_x \left(G' t + \frac{G''}{\omega_y}\right), \tag{52}$$

where $q_x = P_x \left(1 - \frac{5x}{7R}\right)$, $u_x = P_x \frac{x}{R} \left(\frac{5}{7} V_{0y} + R\omega_{zi}\right)$. Taking in account Eqs (2), (8) and (52) we get the next differential equation

$$m\ddot{y}_s + \frac{7G''}{2\omega_y} q_x \dot{y}_s + \frac{7}{2} G' q_x y_s = -u_x \frac{7}{2} \left(G' t + \frac{G''}{\omega_y}\right) \tag{53}$$

As we can see, this equation is the non-homogeneous differential equation of second order with variable coefficients, depending of the normal displacement *x* and the time *t*.
For practical application of the differential equation (53) with the variable viscoelasticity coefficients, we can find their approximate solutions in the same manner as for the equations with the equivalent constant viscoelasticity coefficients (parameters). We can lead the equation (53) to the equation with the constant viscoelasticity parameters as

$$m\ddot{y}_s + B_s \dot{y}_s + C_s y_s = -f_s(t), \tag{54}$$

where $f_s(t)$ is the perturbing force.
The constant parameters B_y, C_y can be found again by using the method equivalent works [49, 50, 51], by finding the works A_{ycm} and A_{ybm} with the variable viscoelasticity parameters equal to the works with constant parameters. Since, according to the boundary conditions $t = \tau_1$, $x = x_m$, $y_s = y_{sm}$ and $q_x = q_{xm}$, $u_x = u_{uxm}$, we can write that

$$A_{ycm} = C_s \int_0^{y_{sm}} y_s dy_s = \frac{1}{2} C_s y_{sm}^2, \tag{55}$$

on the other hand

$$A_{ycm} = \frac{7}{2} G' \int_0^{y_{sm}} q_x y_s dy_s = \frac{7}{4} G' q_{xm} y_{sm}^2, \tag{56}$$

hence, it follows that

$$C_s = \frac{7}{2} G' q_{xm}, \tag{57}$$

where $q_{xm} = P_{xm}\left(1 - \frac{5x_m}{7R}\right)$and where $P_{xm} = \left(k_h x_m + 2k_p R^{\frac{1}{2}} x_m^{\frac{1}{2}}\right)$, and also we can write

$$A_{ybm} = B_s \int_0^{y_{sm}} \dot{y}_s dy_s = B_s \frac{\int_0^{y_{sm}} \int dy_s dy_s}{\int_0^{\tau_1} dt} = B_s \frac{y_{sm}^2}{2\tau_1} \tag{58}$$

and

$$A_{ybm} = \frac{7G''}{2\omega_y} \int_0^{y_{sm}} q_x \dot{y}_s dy = \frac{7G''}{2\omega_y} q_{xm} \frac{y_{sm}^2}{2\tau_1}, \tag{59}$$

and as well it follows that

$$B_s = \frac{7G''}{2\omega_y} q_{xm} \tag{60}$$

Since the function $f_s(t) = u_x \frac{7}{2}(G't + \frac{G''}{\omega_y}) = P_x \frac{7x}{2R}\left(\frac{5}{7}V_{0y} + R\omega_{zi}\right)(G't + \frac{G''}{\omega_y})$ in the initial time $t=0$, $x=0$, $y_s=0$ and at the time of the end of impact $t = \tau_x$ equals zero, because the functions $P_x = k_h x + 2k_p R^{1/2} x^{1/2} = k_h \frac{V_x}{\omega_x} e^{-\delta_x t} sin(\omega_x t) + 2k_p R^{1/2}\left(\frac{V_x}{\omega_x} e^{-\delta_x t} sin(\omega_x t)\right)^{\frac{1}{2}}$ and $x = \frac{V_x}{\omega_x} e^{-\delta_x t} sin(\omega_x t)$ in the initial time $t=0$, $x=0$, $y_s = 0$ and at the time of the end of impact $t = \tau_x$ equal zero as well, and also according to the boundary value problem $t = \tau_1$, $x = x_m$, $y_s = y_{sm}$, $u_x = u_{xm}$, we can take the approximating function of the perturbing force as $f_s(t) = F_s \sin(\omega_x t)$, which obviously should satisfy these boundary value problems. Now, the constant force F_s can be found by using again the method of the equivalent works. It is obvious, we can take that

$$\int_0^{y_{sm}} F_s \sin(\omega_x t)\, dy_s = \int_0^{y_{sm}} u_x \frac{7}{2}(G't + \frac{G''}{\omega_y}) dy_s, \tag{61}$$

Thus, after the integration of Eq. (61), we get

$$F_s = u_{xm} \frac{7}{2\sin(\omega_x \tau_1)}(G'\tau_1 + \frac{G''}{\omega_y}), \tag{62}$$

where $u_{xm} = P_{xm} \frac{x_m}{R}\left(\frac{5}{7}V_{0y} + R\omega_{zi}\right)$.
Thus, the equation (54) can be written as the simple equation with constant coefficients, as follows

$$m\ddot{y}_s + B_s \dot{y}_s + C_s y_s = -F_s \sin(\omega_x t) \tag{63}$$

The solution of the Eq (63) can be found as sum $y_s = y_{s1} + y_{s2}$, where the solution for ys1is known as

$$y_{s1} = e^{-\delta_s t}(C_1 \cos(\omega_s t) + C_2 sin(\omega_s t)), \tag{64}$$

where: $\omega_s = \omega_y = \sqrt{\omega_{0s}^2 - \delta_s^2}$; $\delta_s = \frac{B_s}{2m}$is the tangential damping factor; $\omega_{0s} = \sqrt{\frac{C_s}{m}}$ is the angular frequency of the harmonic oscillations by axis Y.
The solution for $f_s(t) = F_s \sin(\omega_x t)$, can be obtained by the method of undetermined coefficients as:

$$y_{s2} = Acos(\omega_x t) + B\, sin(\omega_x t), \tag{65}$$

where, taking in account that $\delta_s = \frac{B_s}{2m}$ and $\omega_{0s}^2 = \frac{C_s}{m}$, the coefficients A and B have been found respectively as:

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$$A = \frac{2F_s\delta_s\omega_x}{m[(\omega_{0s}^2-\omega_x^2)^2+4\delta_s^2\omega_x^2]}, \tag{66}$$

$$B = -\frac{F_s(\omega_{0s}^2-\omega_x^2)}{m[(\omega_{0s}^2-\omega_x^2)^2+4\delta_s^2\omega_x^2]} \tag{67}$$

Thus, we get the general solution in the next view:

$$y_s = e^{-\delta_s t}(C_1\cos(\omega_s t) + C_2 sin(\omega_s t)) + A cos(\omega_x t) + B\,sin(\omega_x t) \tag{68}$$

Further, according with initial conditions $t = 0$, $y(0) = 0, y_s(0) = 0$ we get the constant of integration $C_1 = -A$ and since $\dot{y}_s(0) = V_{0y}$, it follows that the constant of integration
$C_2 = C = \frac{V_{0y}-A\delta_s-B\omega_x}{\omega_s}$, and respectively we get

$$y_s = Ce^{-\delta_s t}sin(\omega_s t) + A[\cos(\omega_x t) - e^{-\delta_s t}cos(\omega_s t)] + B\,sin(\omega_x t) \tag{69}$$

and then as well after the differentiation it follows that

$$\dot{y}_s = Ce^{-\delta_s t}[\omega_s\cos(\omega_s t) - \delta_s sin(\omega_s t)] + \\ +A\{e^{-\delta_s t}[\omega_s\sin(\omega_x t) + \delta_s cos(\omega_s t)] - \omega_x sin(\omega_x t)\} + B\omega_x cos(\omega_x t) \tag{70}$$

Finally, by using Eqs (18), (14), we get

$$y = \frac{5}{7}V_{0y}t + R\omega_{zi}t + \\ +\frac{2}{7}\{Ce^{-\delta_s t}sin(\omega_s t) + A[\cos(\omega_x t) - e^{-\delta_s t}cos(\omega_s t)] + B\,sin(\omega_x t)\} \tag{71}$$

and

$$\dot{y} = \frac{5}{7}V_{0y} + R\omega_{zi} + \frac{2}{7}\{Ce^{-\delta_s t}[\omega_s\cos(\omega_s t) - \delta_s sin(\omega_s t)] + \\ +A\{e^{-\delta_s t}[\omega_s\sin(\omega_x t) + \delta_s cos(\omega_s t)] - \omega_x sin(\omega_x t)\} + B\omega_x cos(\omega_x t)\}, \tag{72}$$

and also by using Eqs (20) and (16), we obtain

$$y_\varphi = \frac{5}{7}V_{0y}t + R\omega_{zi}t - \\ -\frac{5}{7}\{Ce^{-\delta_s t}sin(\omega_s t) + A[\cos(\omega_x t) - e^{-\delta_s t}cos(\omega_s t)] + B\,sin(\omega_x t)\}, \tag{73}$$

and

$$\dot{y}_\varphi = \frac{5}{7}V_{0y} + R\omega_{zi} - \frac{5}{7}\{Ce^{-\delta_s t}[\omega_s\cos(\omega_s t) - \delta_s sin(\omega_s t)] + \\ +A\{e^{-\delta_s t}[\omega_s\sin(\omega_x t) + \delta_s cos(\omega_s t)] - \omega_x sin(\omega_x t)\} + B\omega_x cos(\omega_x t)\} \tag{74}$$

4. BORDER OF APPLICATION OF THE OBTAINED SOLUTIONS

The obtained results for viscoelastic displacement at impact have the borders of application, which can be found by using the next two equations, such as:

$$\omega_x = \sqrt{\omega_{0x}^2 - \delta_x^2} \tag{75}$$

and

$$\omega_s = \sqrt{\omega_{0s}^2 - \delta_s^2}, \tag{76}$$

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First of all, since as $\omega_{0x}^2 = \frac{C_x}{m}$ and $\delta_x = \frac{B_x}{2m}$, follows $\delta_x = \frac{B_x \omega_{0x}^2}{2C_x}$, then, taking in account Eqs. (31), (32), and since $tg\beta_E = \frac{E''}{E'}$, as well it follows that $\delta_x = \frac{3\omega_{0x}^2}{2\omega_x} tg\beta_E$. After substituting this ratio into Eq. (75) we get the next algebraic equation

$$\omega_x^4 - \omega_{0x}^2 \omega_x^2 + \frac{9}{4} \omega_{0x}^4 tg^2 \beta_E = 0 \tag{77}$$

This equation has only the one valid solution

$$\omega_x^2 = \frac{\omega_{0x}^2}{2} \left(1 + \sqrt{1 - 9tg^2 \beta_E}\right) \tag{78}$$

and it has the valid root only when$1 - 9tg^2 \beta_E \geq 0$, therefore for the compression we get

$$tg\beta_E = \frac{E''}{E'} \leq \frac{1}{3} \tag{79}$$

and according to Eq. (39′) we get for a viscoelastic contact that

$$k_x \geq \frac{\pi - 1}{\pi + 1} \tag{80}$$

It is obvious that, when $k_x < \frac{\pi - 1}{\pi + 1}$ the plastic deformations will begin to act in the zone of the contact.

On the other hand, since $\delta_s = \frac{B_s}{2m}$, $\omega_{0s}^2 = \frac{C_s}{m}$ and $tg\beta_G = \frac{G''}{G'}$, taking in account Eqs. (57) and (60), it follows $\delta_s = \frac{\omega_{0s}^2}{2\omega_s} tg\beta_G$, and respectively we get the next algebraic equation

$$\omega_s^4 - \omega_{0s}^2 \omega_s^2 + \frac{1}{4} \omega_{0s}^4 tg^2 \beta_G = 0 \tag{81}$$

This equation has only one valid solution

$$\omega_s^2 = \frac{\omega_{0s}^2}{2} \left(1 + \sqrt{1 - tg^2 \beta_G}\right) \tag{82}$$

and it has the valid root only when$1 - tg^2 \beta_G \geq 0$, therefore for the rolling shear we get

$$tg\beta_G = \frac{G''}{G'} \leq 1 . \tag{83}$$

5. EXAMPLE AND CONCLUSION

Let us consider, for example, the collision between the steel ball, having the radius R =10 mm, and a high-elastic semi-space of elastomer. Also let the velocities of centre of mass of the ball are: $V_{0x} = 1.0 \left(\frac{m}{c}\right)$; $V_{0y} = 1.0 \left(\frac{m}{c}\right)$; $\omega_{zi} = 0$. Also, we can take $k_p = \sqrt{2}$, $k_h = 1$. The Poisson's coefficient for elastomer can be taken $\nu = 0.5$, and let the tangent of mechanical losses $tg\beta_G = tg\beta_E = 0.2$. and $E' = 2 \times 10^{-6}$ (Pa). Since the density of steel ρ = 7.8 ×103 (kg/m^3), the mass of ball has been found as $m = \frac{4}{3}\pi R^3 \rho = 32.67 \times 10^{-3}$(kg). Then, after calculations, using equations (39) and (41) we got for the restitution coefficient $k_x = 0.68$ and for the maximum magnitude of compression (indentation) we got respectively $x_m = 1.4\ (mm)$. Also, since $\omega_{0x} = \sqrt{\frac{C_x}{m}}$ and $\omega_{0s} = \sqrt{\frac{C_s}{m}}$, $q_{xm} = P_{xm}\left(1 - \frac{5x_m}{7R}\right)$, using Eqs. (31), (57) and (75), (76) we got $\omega_x = 563.6$, $\left(\frac{Rad}{s}\right)$ and

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$\omega_s = 881\ (\frac{Rad}{s})$. Then, the time of the maximum indentation (compression) has been calculated as $\tau_x = \frac{\pi}{\omega_x} = 5.855\ (ms)$. Then further, since $\delta_x = \frac{B_x}{2m}$ and $\delta_s = \frac{B_s}{2m}$, and as well, since $u_{xm} = P_{xm}\frac{x_m}{R}\left(\frac{5}{7}V_{0y} + R\omega_{zi}\right)$, $C = \frac{V_{0y} - A\delta_s - B\omega_x}{\omega_s}$, using Eqs. (62), (66), (69) (71) and (73) the graphs of motion of centre of mass of the ball have been built as it is represented in the Figure 3.

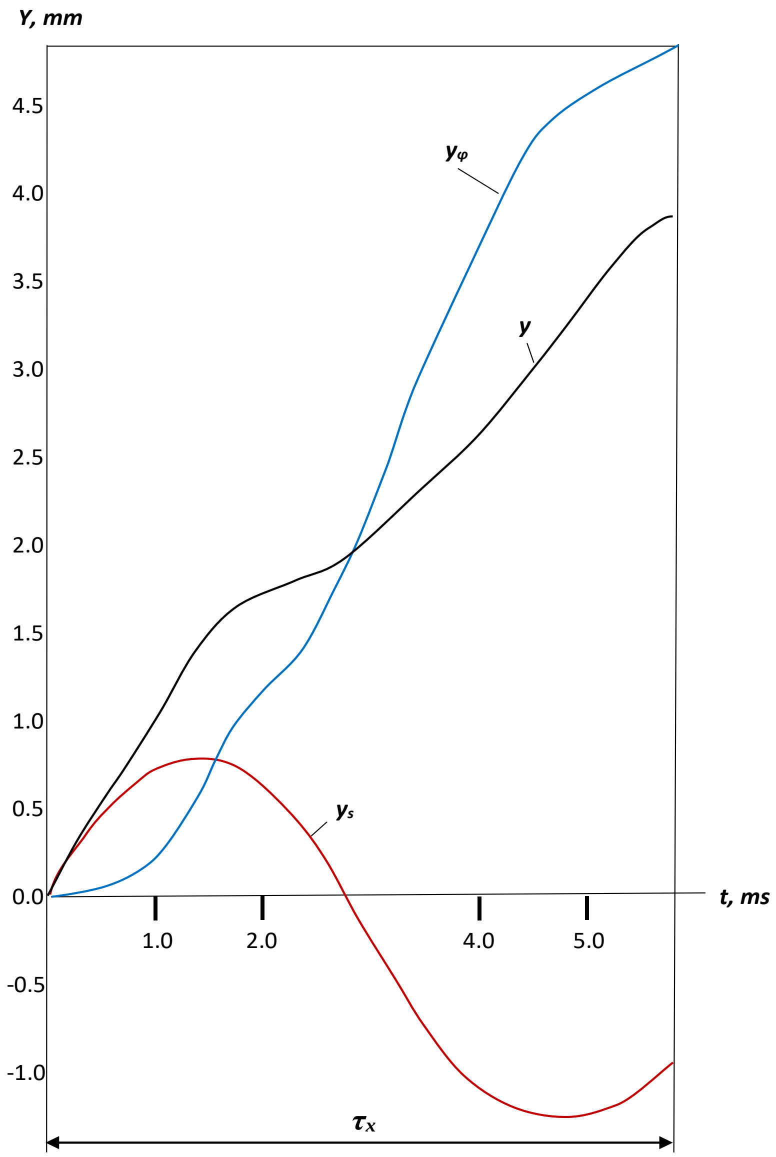

Figure 3. The graphs of motion of the centre of mass of the ball

The displacements for some specific points of the time at impact have been calculated and are given here below:

- For the moment of the time $t = \pi/4\omega_s$ = 0.89 (ms) respectively y_s= 0.674(mm); y_φ= 0.154 (mm); y = 0.828 (mm).
- For the moment of the time $t = \pi/2\omega_s$ = 1.763 (ms) respectively y_s = 0.72(mm); y_φ = 1.063 (mm); y = 1.183 (mm).

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- For the moment of the time $t = \tau_1$ = 2.37 (ms) respectively y_s= 0.351(mm); y_φ= 1.44 (mm); y = 1.791 (mm).
- For the moment of the time $t = \tau_s$ = 3.566 (ms) respectively y_s = -0.792 (mm); y_φ= 3.12 (mm); y = 3.041 (mm).
- For the moment of the time $t = \tau_x$ = 5.86 (ms) respectively y_s = - 0.95 (mm); $y\varphi$ = 4.864 (mm); y = 3.914 (mm).

In conclusion, first of all, let us mark here, that the obtained equations for displacement (motion) of the centre of mass of a body at impact can be useful for finding all displacements and velocities during all the time of moving and in the instant of rebound $t = \tau_x$. The obtained results, for example, can be used in the design of wear-resistant elements and coverings for components of machines and equipment, which are working in harsh conditions where they are subjected to the action of flow or jet rigid particles. Also, this article can be useful to help the determination of contact stresses, durability and fatigue life for wide spectrum of tasks relevant to collisions between solid bodies under different loading conditions. Opportunities exist to use this theory practically, for example, in the design and development of new advanced materials, wear-resistant elastic coatings and elements for pneumatic and hydraulic systems, stop valves, fans, centrifugal pumps, injectors, valves, gate valves and in other installations.

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